# ANR Distancia
# Paris 2019 Meeting
# Open Problem Session

27 March 2019

1. (Pierre Charbit) This problem is a subpart of our project ANR Distancia. When given an hypergraph $\mathcal{H}$ we denote by $\tau$ the minimum size of an hitting set - a set of vertices that intersects every hyperedge. Dually, one denotes by $\nu$ the maximum size of a packing - a collection of disjoint hyperdeges. It is straightforward that $\tau \geq \nu$. On the other hand, there exists families of hypergraph where $\nu$ is fixed but $\tau$ goes to infinity.

   Given a graph $G = (V, E)$, one can associate to it the hypergraph on ground set $V$ where the hyperdeges are the $|V(G)|$ balls of radius $R$ around each vertex. We denote by $\tau_R(G)$ and $\nu_R(G)$ the minimum hitting set and max packing for this hypergraph.

   The question is to understand for which classes $\mathcal{C}$ of graphs there exists a function $f$ such that for any graph $G$ in the class $\tau_R(G) \leq f(\nu_R(G))$

   Chepoi, Estellon and Vaxes proved the following : there exists a constant $C_{planar}$ (not depending on $R$) such that for every planar graph $G$ of diameter at most $2R$ (or equivalently such that $\nu_R(G) = 1$), $\tau_R(G) \leq C_{planar}$.

   A fantastic result would be to prove a similar result for any of $\nu_R(G)$.

   If one does not ask for $C$ independent of $R$, this is true, as Dvorak proved a more general result : For any nowhere dense class $\mathcal{C}$ (so in particular every minor-closed class, so in particular planar), for any $R$, there exists a constant $C(R, \mathcal{C})$ such that for every graph $G$ in $\mathcal{C}$, $\tau_R(G) \leq C(R, \mathcal{C})\nu_R(G)$.

   It is possible to prove that for nowhere dense class, one cannot hope to have a constant $c$ not dependent on $R$, but no counter example is known for a minor closed class.

2. (Denis Cornaz) A *signed graph* is defined as a pair $(G, R)$, where $G = (V, E)$ is a graph and $E$ is a subset of $E$. Edges in $R$ are called negative edges. An *odd circuit* of $(G, R)$ is a circuit of $G$ containing an odd number of edges in $R$, and a *flow* is a circuit containing exactly one edge of $R$.

   Given a partition $(A, B)$ of $V$, a resiging of $(G, R)$ with respect to this cut is the operation of replacing $R$ by $R \Delta E(A, B)$, where $\Delta$ is the symmetric difference operation and $E(A, B)$ is the set of edges in the cut between $a$ and $B$. Observe that resigning does not change which cycles are odd cycles.

   A signed graph $(H, R')$ is a odd-minor of a signed graph $(G, R)$ if there exists a sequence of resigning, vertex deletion, edge deletion and contraction of *non negative* edges (in any order) going from $(G, R)$ to $(H, R')$.

Let $\tau_{odd}$ (resp. $\tau_1$) denote the size of a minimal set of vertices that hits all odd cycles (resp. flows). Let $\nu_{odd}$ (resp $\nu_1$) denote the size of a maximal collection of disjoint odd cycles (resp. flows).

There exists a characterization of signed graphs such that $\tau_{odd} = \nu_{odd}$ - they are exactly the signed graphs that do not have $(K_4, E(K_4))$ as a minor.

Question : Give a condition for a signed graph to satisfy $\tau_1 = \nu_1$.

3. Pascal Préa

   A *Robinson Dissimilarity* is a integer square symmetric matrix such that for any $i < j < k$, $M_{i,j} < M_{i,k}$ and $M_{i,k} > M_{j,k}$. In other words if one looks at row $i$ the first $i$ entries are decreasing and the last $n - i$ entries are increasing.

   Such a matrix can be obtained by considering the distances of an ordered set of points on a line.

   If the matrices are $0, 1$-matrices they re well studied and understood, and the are in bijection with the so-called Dyck paths.

   Question : Compute the number of Robinson dissimilarites for other set of integers as entries? For example : $(0, 1, 2)$?

4. (Kolja Knauer) For any $d, k, r$, construct a median graph $G$ such that :

   - It has isometric dimension $d$
   - It is $Q_r$-free
   - There exists a vertex $v$ such that $S_k(v)$ (the set of vertices at distance exactly $k$ from $v$ is large.

5. (Michel Habib) Given a family $\mathcal{F}$ of ordered graphs, the hereditary class $LinForb(\mathcal{F})$ is defined as the set of graphs for which there exists an ordering of the nodes, such that no induced ordered subgraph is isomorphic to an ordered graph in $\mathcal{F}$.

   The classical result stating that chordal graphs are characterized by the existence of a perfect elimination order says exactly that the class of chordal graphs is $LinForb(P)$, where $P$ is the induced path on three vertices ordered in such a way that the middle vertex of the past is last in the order. If the 3 vertices are ordered such that the middle vertex is in the middle then forbidding it yields the class of comparability graphs and several classical classes, especially ones corresponding to geometric intersection model can be defined in this fashion.

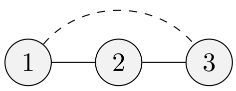
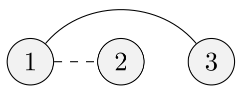



Figure 1: a forbidden ordered graph for comparability graphs

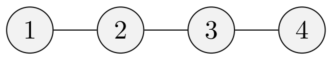
For interval graphs, it is known that they are characterized by forbidding the following two ordered subgraphs

This characterization of interval graphs is not only simple to express but also useful since there exists good recognition algorithm that take advantage of it.

In the example for interval graphs above, we see that the presence of an edge between the two vertices 2 and 3 is irrelevant, so sometimes to simplify things we simply say that we forbid an ordered trigraph, that is a graph with edges, non edges, and undecided edges, meaning that we forbid all ordered subgraphs that are realizations of this trigraphs (a realization of a trigraph is any graph where the undecided edge have been given a status : either edge or non edge). So interval graphs are defined by forbidding the following pattern.

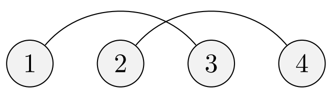



There are many other interesting examples and this is not necessarily well known so let us present two more. Forbidding the pattern below corresponds to 3-colourable graphs. this can easily be generalized to $k$ colourability and even partition into $k$ stable sets and

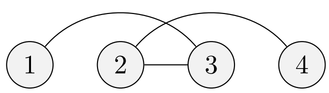



$k'$-cliques.

One can also prove that outerplanar graphs are defined by forbidding the following pattern



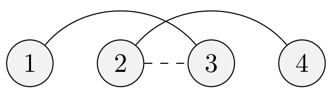

A superclass is the class of $p$-box graphs which are defined as intersection model of rectangles in the plane with sides parallel to the xaxis and yaxis, and whose lower left corner are all on some fixed Line L of the plane. It is known that they are characterized by forbidding the following slightly more precise pattern (it is therefore a super class of outerplanar graphs)




Recognition problems are interesting : it is known that for every family of forbidden ordered subgraphs of size, each of which as at most 3 vertices, the corresponding recognition problem is polynomial.

On the other hand, for patterns on 4 vertices there exists some polynomial (ex: outerplanar) and some NP Hard problems (ex: 3-colourable).

Question : Is there a way to understand which pattern are going to define NP-hard or polynomial recognition problems? What about the pattern below?



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